SONIC GAS FLOW AROUND A NONSYMMETRIC PROFILE

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A particular solution of the transonic equations describing two-dimensional flow of an ideal gas is obtained for nonsymmetric flow around a certain profile. The aerodynamic characteristics of the profile are determined.

In the plane of the velocity hodograph, transonic flow of a gas is described by the Tricomi equation

$$\eta\partial^2\psi/\partial\theta^2 + \partial^2\psi/\partial\eta^2 = 0 \tag{1}$$

for the stream function $\psi(\theta, \eta)$. In the above equation, θ is the slope of the velocity vector with respect to the flow direction at infinity, and η is the velocity modulus function, introduced by F. I. Frankl' [1], which is either positive or negative for subsonic or supersonic velocities, respectively. We use the following linear combination of self-similar solutions as the solution of Eq. (1):

$$\begin{aligned}\psi(\theta, \eta) &= c_1\psi_1(\theta, \eta) + c_2\psi_2(\theta, \eta) + c_3\psi_3(\theta, \eta) + c_4 = \\ &= c_1\rho^{-5/3}f_{-5/3}(t) + c_2\rho^{1/3}f_{1/3}(t) + c_3\rho^{-2/3}f_{-2/3}(t) + c_4,\end{aligned} \tag{2}$$

where

$$f_{-5/3}(t) = 9^{-1}(3/2)^{1/3}[(1-t)^{1/3}(1+3t) - (1+t)^{1/3}(1-3t)];$$

$$f_{1/3}(t) = (3/2)^{1/3}[(1-t)^{1/3} - (1+t)^{1/3}];$$

$$f_{-2/3}(t) = -3^{-1}(3/2)^{1/3}[(1-t)^{1/3} + (1+t)^{1/3}],\ \rho^2 = \theta^2 + (4/9)\eta^3,$$

$$t = \theta/\rho.$$

Here c_1, c_2, and c_3 are arbitrary constants. The constant c_4 is determined below. The first two solutions in (2) were investigated in [1, 2]; they describe the gas flow for $M_\infty = 1$ at a point remote from the solid and in the neighborhood of the Laval nozzle's center, respectively. The third solution, in combination with the first, describes the flow behavior in approaching infinity for nonsymmetric flow around the profile. In order to determine the shape of the solid in the flow plane, we must find the zero streamline in the hodograph plane and then calculate the x coordinates for it. The shape of the circumfluous profile is then obtained by integrating with respect to x the slope of the zero streamline. We use the differential relationship of the stream function

$$\psi_1(\theta, \eta) = \partial^2\psi_2(\theta, \eta)/\partial\theta^2,\ \psi_3(\theta, \eta) = \partial\psi_2(\theta, \eta)/\partial\theta \tag{3}$$

and the cubic function [3], which is satisfied by the stream function $\psi_2(\theta, \eta)$:

$$\psi_2^3 + 3\eta\psi_2 + 3\theta = 0. \tag{4}$$

By differentiating (4) with respect to θ and using relationship (3), we obtain

$$\psi_1(\theta, \eta) = -2\psi_2(\psi_2^2 + \eta)^{-3},\quad \psi_3(\theta, \eta) = -(\psi_2^2 + \eta)^{-1}. \tag{5}$$

We substitute (5) in (2) and set equal to zero the obtained expression

$$\begin{aligned}b\psi_2[a - (\psi_2^2 + \eta)][a^2 + a(\psi_2^2 + \eta) + (\psi_2^2 + \eta)^2] + \\ + (\psi_2^2 + \eta)[c_3/c_4 - (\psi_2^2 + \eta)] = 0,\end{aligned} \tag{6}$$

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where $a^3 = 2c_1/c_2$ and $b = c_2/c_4$. If we now put in (6)

$$c_3/c_4 = a = (2c_1/c_2)^{1/3}, \tag{7}$$

we obtain from (6) and (4) the implicit equation of the contour of the nonsymmetric profile in parametric form in the hodograph plane:

$$a^2(3b\psi_2 + 1) = 0, \quad \theta = -3^{-1}\psi_2(a + 2\eta), \quad \psi_2^2 + \eta = a. \tag{8}$$

The parameter in (8) is the function $\psi_2(\theta, \eta)$. Expression (7) defines the constant c_4 in terms of the arbitrary constants c_1, c_2, and c_3. By solving simultaneously Eqs. (8), we find the coordinates of the leading point B of the profile in the hodograph plane,

$$\eta_B = a - 1/(9b^2), \; \theta_B = [3a - 2/(9b^2)]/(9b). \tag{9}$$

For passage to the flow plane, we use the transonic relationship [4] $x = (\varkappa + 1)^{1/3}\varphi(\theta, \eta)$ and then represent the x coordinate of the zero streamline in the following form:

$$(\varkappa + 1)^{-1/3}x = c_1\varphi_1(\theta, \eta) + c_2\varphi_2(\theta, \eta) + c_3\varphi_3(\theta, \eta). \tag{10}$$

The velocity potential $\varphi_n(\theta, \eta)$ of the self-similar solution is related to the function $f_n(t)$ by the expression [4]

$$\varphi_n(\theta, \eta) = \rho^{n+1/3}g_n(t) = (3/2)^{1/3}\rho^{n+1/3}(1 - t^2)^{2/3}f_n'(t)/(n + 1/3),$$

by means of which we find

$$\varphi_1(\theta, \eta) = (\psi_2^2 - \eta)(\psi_2^2 + \eta)^{-3}, \quad \varphi_2(\theta, \eta) = -2^{-1}(\psi_2^2 + 2\eta), \tag{11}$$
$$\varphi_3(\theta, \eta) = \psi_2(\psi_2^2 + \eta)^{-1}.$$

By substituting (11) in (10) and considering the equation $\psi_2^2 + \eta = a$ from (8), we obtain the relationship

$$x = (\varkappa + 1)^{1/3}c_2[-(3/2)\eta \pm ca^{-1}(a - \eta)^{1/2}], \tag{12}$$

where $c = c_3/c_2$, while the plus and minus signs pertain to the lower and upper parts of the nonsymmetric profile, respectively. Equation (12) indicates that the velocity is distributed nonlinearly over the profile's surface. We find the shape of the circumfluous body in the flow plane by calculating the integral

$$y(x) = \int_a^\eta \theta dx. \tag{13}$$

As a result of calculating integral (13), we obtain the following parametric equation of the upper and lower parts of the nonsymmetric profile in terms of physical variables:

$$(\varkappa + 1)^{-1/3}y = \mp(2/5)c_2(a - \eta)^{5/2} + (3a)^{-1}c\eta^2 \pm c_2a(a - \eta)^{3/2} + \tag{14}$$
$$+ 3^{-1}c\eta - (2/3)ac,$$
$$(\varkappa + 1)^{-1/3}x = c_2[-(3/2)\eta \mp ca^{-1}(a - \eta)^{1/2}].$$

After eliminating the parameter η in (14), we write the equation of the nonsymmetric profile in explicit form:

$$(\varkappa + 1)^{-1/3}y(x) = \mp 4(15a_1)^{-1}[a_2x + 2a_3(a_4 + a_2x)^{1/2} + a_5]^{5/2} + \tag{15}$$
$$+ a_3[a_2x + 2a_3(a_4 + a_2x)^{1/2} + 2a_3^2] \pm 2(3a_1)^{-1}a[a_2x + 2a_3(a_4 + a_2x)^{1/2} +$$
$$+ a_5]^{3/2} - aa_3[a_2x + 2a_3(a_4 + a_2x)^{1/2} + 2a_3^2] - a_6,$$

where $a_1 = 2(3c_2)^{-1}$, $a_2 = (\varkappa + 1)^{-1/3}a_1$, $a_3 = (3a)^{-1}c$, $a_4 = a + a_3$ and $a_5 = a + 2a_3^2$, $a_6 = 2\cdot 3^{-1}ac$.

Assuming that $c_3 = c_4 = 0$ in (15), we obtain the contour equation for Guderley's symmetric profile [4]

$$y(x) = 5^{-1}(\varkappa + 1)^{1/3}c_2(a_2x + a)^{3/2}(3a - 2a_2x). \tag{16}$$

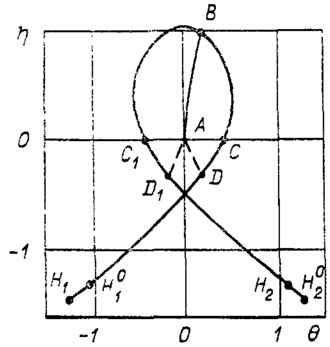

Fig. 1

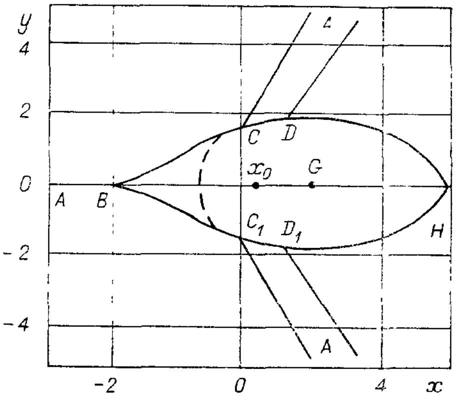

Fig. 2

For convenience in constructing the nonsymmetric profile in the hodograph plane and the physical plane, we put

$$c_1 = c_2 = 1,\ c_3 = c_4 = 2b_1,\ \psi_2 = f,\ z = f^2 + \eta. \tag{17}$$

Figure 1 shows the flow hodograph, plotted for the cases $b_1 = 0;\ -0.1$. It should be noted that the shape of the circumfluous profile in the hodograph plane does not depend on the b_1 parameter. At the same time, for different b_1 values, the zero streamline AB arrives at different points of the profile, determined by (9). If the thus obtained contour is considered as the region boundary for the Tricomi problem and the boundary condition $\psi = 0$, then, as is known [5], the problem is correct in the region bounded by the ABCD contour and the line AD and, correspondingly, in the ABC_1D_1A region. The curves AD and AD_1 of Eq. (1) start at the profile and arrive at the point A corresponding to infinity in the flow plane, so that the flow disturbances introduced by the part of the profile located aft of the points D and D_1 do not affect the flow in the AD_1C_1BCDA region. Consequently, the profile aft of the points D and D_1 can be assigned arbitrarily without altering the flow in the region up to the limiting curves AD and AD_1. It should be noted here that, along the AD and AD_1 curves, the discriminant $Q = \theta^2 + (4/9)\eta^3$ of Eq. (4) vanishes, having a positive value upstream, and a negative value downstream. Thus, formal continuation of solution (2) into the region with $Q < 0$ leads to three-valued results in correspondence with the roots of Eq. (4):

$$f_1 = 2(-\eta)^{1/2}\cos(\beta/3),\ f_{2/3} = -2(-\eta)^{1/2}\cos(\beta/3 + \pi/3),\ \cos\beta = (3/2)\theta(-\eta)^{-3/2}.$$

However, analytical continuation of the solution through AD_1 yields only the root f_1, while continuation through AD yields the root f_2. It should also be noted that these continuations do not have limiting curves, i.e., they allow one-to-one transition to the physical coordinates x, y.

If the profile is to be closed in the physical plane, it is necessary that the end points of the contour continuations in the hodograph plane correspond to the same point in the physical plane. With an allowance for (17), the parametric equation of the profile in the physical plane assumes the following form:

$$\begin{gathered} y = (\varkappa + 1)^{1/3} f^2 [b_1(1 - 3^{-1}f^2) + 2f(2f^2 - 5)/5], \\ x = (\varkappa + 1)^{1/3}[3(f^2 - 1) - 2b_1 f]. \end{gathered} \tag{18}$$

Let us determine the coordinates of the profile's stern point. The conditions $x(f_1) = x(f_2)$ and $y(f_1) = y(f_2)$ must be satisfied at this point. The latter relationships constitute a system of nonlinear algebraic equations with the unknowns f_1 and f_2, which can be transformed to obtain

$$\begin{gathered} (f_2 - f_1)(3h - b_1) = 0,\ h = f_1 + f_2, \\ g^2 + (6h^2 - 5)g - 2h^2(8h^2 - 5) = 0,\quad g = f_1^2 + f_2^2, \end{gathered}$$

whence

$$g = 5 - 8h^2 \quad \text{or} \quad g = 2h^2. \tag{19}$$

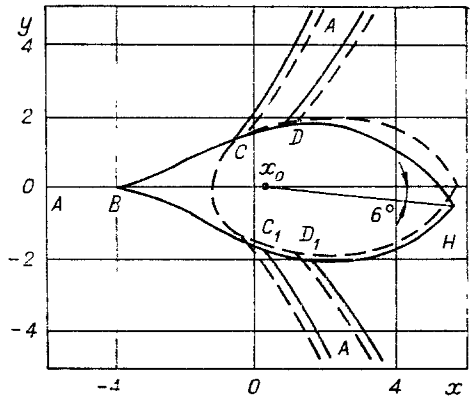

Fig. 3

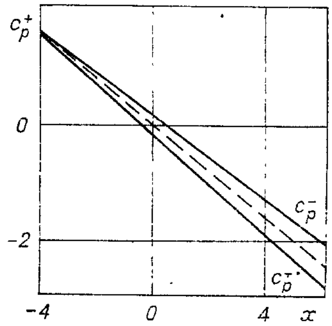

Fig. 4

Since $f_1 \neq f_2$ at the stern point, we find $h = b_1/3$. Then, we have from the first of Eqs. (19)

$$f_{1,2} = b_1/3 \pm \nu, \quad \nu = \sqrt{(5/2)\left[1 - (2/9)\, b_1^2\right]}.$$

The second equation in (19) does not have real roots. Thus, for the coordinates x, y of the stern point H, we obtain

$$x_H = -(\varkappa + 1)^{1/3}(3 + b_1^2/3), \quad y_H = (\varkappa + 1)^{1/3} b_1 \left[15/4 - (7/3)\, b_1 + (1114/81)\, b_1^4\right]. \tag{20}$$

In the hodograph plane, the point H corresponds to the two points H_1 and H_2 with the coordinates

$$\theta_{1,2} = 3^{-1} f_{1,2}(3 - 2f_{1,2}^2), \quad \eta_{1,2} = 1 - f_{1,2}^2.$$

The points H_1^0, H_2^0, H_1, and H_2 for the cases $b_1 = 0;\ -0.1$ are plotted in Fig. 1. In approaching the stern point along the upper and lower sides of the profile, the corresponding velocity vectors have vertical components in opposition to each other, which leads to the development of tail shock waves, which, however, do not affect the flow along the profile boundary. Figures 2 and 3 show the profiles in the physical plane for the cases $b_1 = 0;\ -0.1$, the sonic lines CA, C_1A ($\eta = 0$), and the limiting curves DA, D_1A ($\theta \mp (2/3)(-\eta)^{3/2} = 0$), which were plotted on the basis of the relationships [4] $y = \psi$ and $x = (\varkappa + 1)^{1/3}\varphi$. In the case of $b_1 = 0$, the solution obtained describes the flow around a symmetric profile (16). As b_1 varies, the profile is deformed. For $|b_1| \leqslant 0.1$, the forepart (≈40% of the length) remains virtually unchanged with the exception of a small area around the leading point (≈2% of the length). The shape of the afterbody of the profile (≈60% of the length) remains virtually unchanged, however; it rotates through a certain angle α relative to the direction of the oncoming flow. For $b_1 = -0.1$, we have $\alpha = 6$ deg. For practical purposes, the overall area of the profile remains unchanged. Let us find the trajectory of the stern point in a coordinate system bound to the forepart of the profile. The coordinates of the leading point B are obtained for $f = b_1/3$ by means of relationships (18):

$$x_B = -(\varkappa + 1)^{1/3}(3 + b_1^2/3), \quad y_B = 0.$$

Thus, the displacement Δx_B of the nose point with variation in b_1 is determined by the expression $\Delta x_B = b_1^2/3$. Then, by shifting the origin of the x, y coordinate system Δx_B, we obtain the following expression for the coordinate x of the stern point H:

$$x_H = (\varkappa + 1)^{1/3}\left[9/2 + (5/3)\, b_1^2\right]. \tag{21}$$

Here, the coordinate origin coincides with the projection of the profile's sonic point on the axis for $b_1 = 0$. By writing the equation of motion of the stern point in the form $y^2 + (x - x_0)^2 = R^2$, we obtain on the basis of (21) and the second relationship in (20)

$$x_0 = (9/32)(\varkappa + 1)^{1/3}, \quad R = R_0 + O(b_1^3), \quad R_0 = (135/32)(\varkappa + 1)^{1/3}.$$

The values of x_0 and R_0 determine the circle tangent to the trajectory of the stern point.

Let us determine the aerodynamic characteristics of the profile. As is known [4], the pressure reaction c_p at a point of the profile is determined by the expression $c_p = 2(\varkappa + 1)^{-1/3}\eta$. Figure 4 shows the distribution of $c_p^{\pm}$ values along the profile chord for the upper and lower sides for $b_1 = -0.1$ in comparison with the linear distribution $c_p = -(2/3)(\varkappa + 1)^{-2/3}x$ in the case of a symmetric profile and $b_1 = 0$. Using the expressions

$$c_y = l^{-1}\oint\limits_L c_p dx, \quad c_x = -l^{-1}\oint\limits_L c_p dy, \quad c_M = l^{-2}\oint\limits_L (x - x_1)\, c_p dx,$$

we calculate the coefficient of lift c_y, drag c_x, and moment c_M relative to the point $x = x_1$, $y = 0$. Here, l is the length of the profile chord: $l = x_H - x_B$, and L is the profile boundary. After performing the calculations, we obtain

$$c_y = (8/9)\sqrt{10}\,(\varkappa + 1)^{-1/3}\, b_1 \left[1 - (2/9)\, b_1^2\right]^{1/2} = \sqrt{10}\,(\varkappa + 1)^{-1/3}\,\alpha + O(\alpha^2),$$

$$c_x = (8/21)\sqrt{10}(\varkappa + 1)^{-1/3} \text{ for } b_1 = 0,$$

$$c_M = (4/3)^3\, 0.04\,(\varkappa + 1)^{-2/3}\, b_1 \left[x_1 \nu^3 + 3\,(\varkappa + 1)^{1/3}\left(\nu^2 - 3\nu^5/5 + b_1^2/9\right)\right].$$

We now find the center of pressure G, i.e., the point at which the lift acts. For this, we put $c_M = 0$, whence

$$x_1 = 3\,(\varkappa + 1)^{1/3}\nu^{-3}\left[(3/5)\nu^5 - \nu^2 - b_1^2/9\right] = 1.5\,(\varkappa + 1)^{1/3} + O\left(b_1^2\right).$$

In conclusion, it should be noted that all the above calculations have been performed for a profile with the relative thickness $\tau_0 = (8/25)\sqrt{1.5} \approx 0.39$, $\varkappa = 1.4$. In order to generalize these results and extend them to profiles with an arbitrary relative thickness τ, all the aerodynamic coefficients must be additionally multiplied by $(\tau/\tau_0)^{2/3}$ on the basis of the aerodynamic similarity law for transonic gas flow. Moreover, the law of transonic stabilization makes it possible to extend the above results to cases of high subsonic velocity of the oncoming gas flow.

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